Machine Learning Guidance for an Automatic Theorem Prover

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- The proxy (ML) task vs the true target

Machine-Learning-Boosted Automated Theorem Proving

ATP technology: saturation-based

- state of the art (cf. CASC)
- E, iProver, SPASS, Vampire, ...

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Three main contributions:

- a RL-inspired learning operator
- a new neural architecture (GNN + RvNNs + MLP)
- 20 % performance boost of Vampire under neural guidance

Outline

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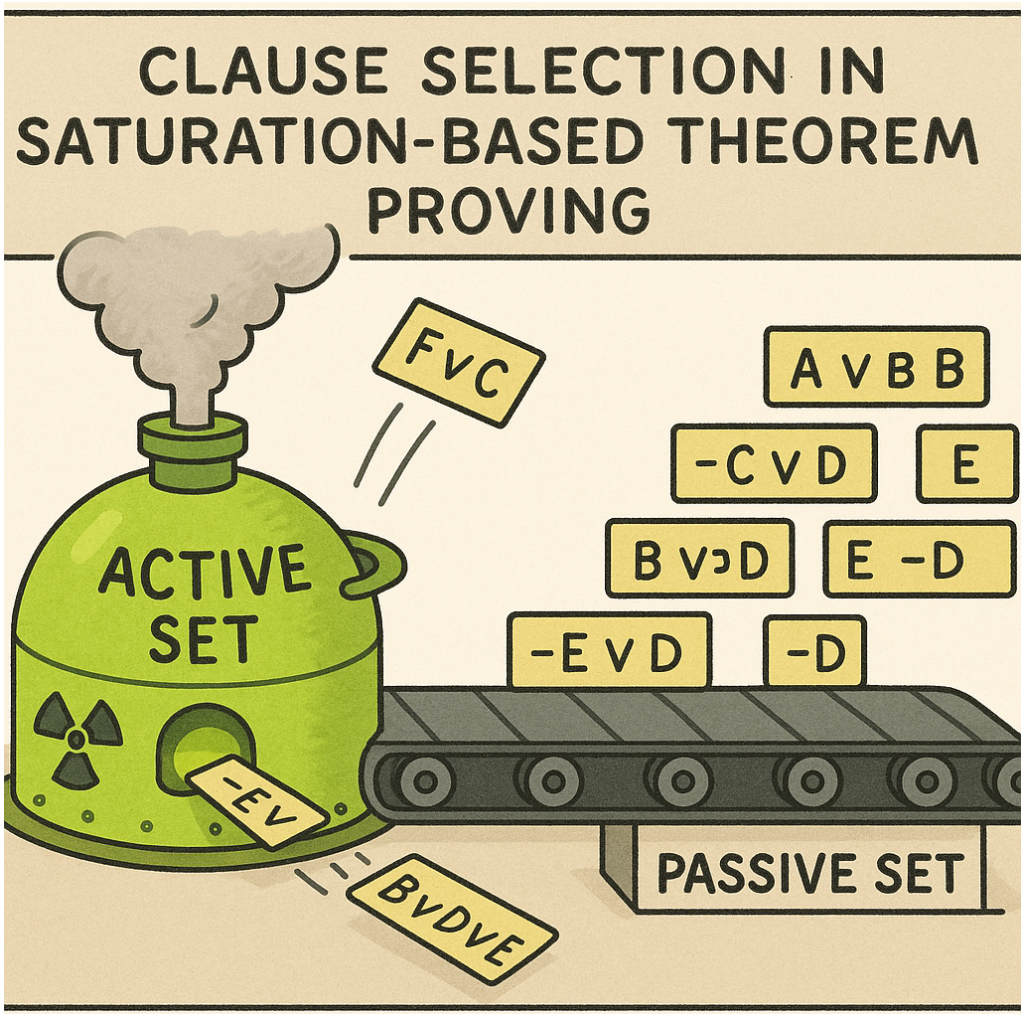
Saturation-based Theorem Proving



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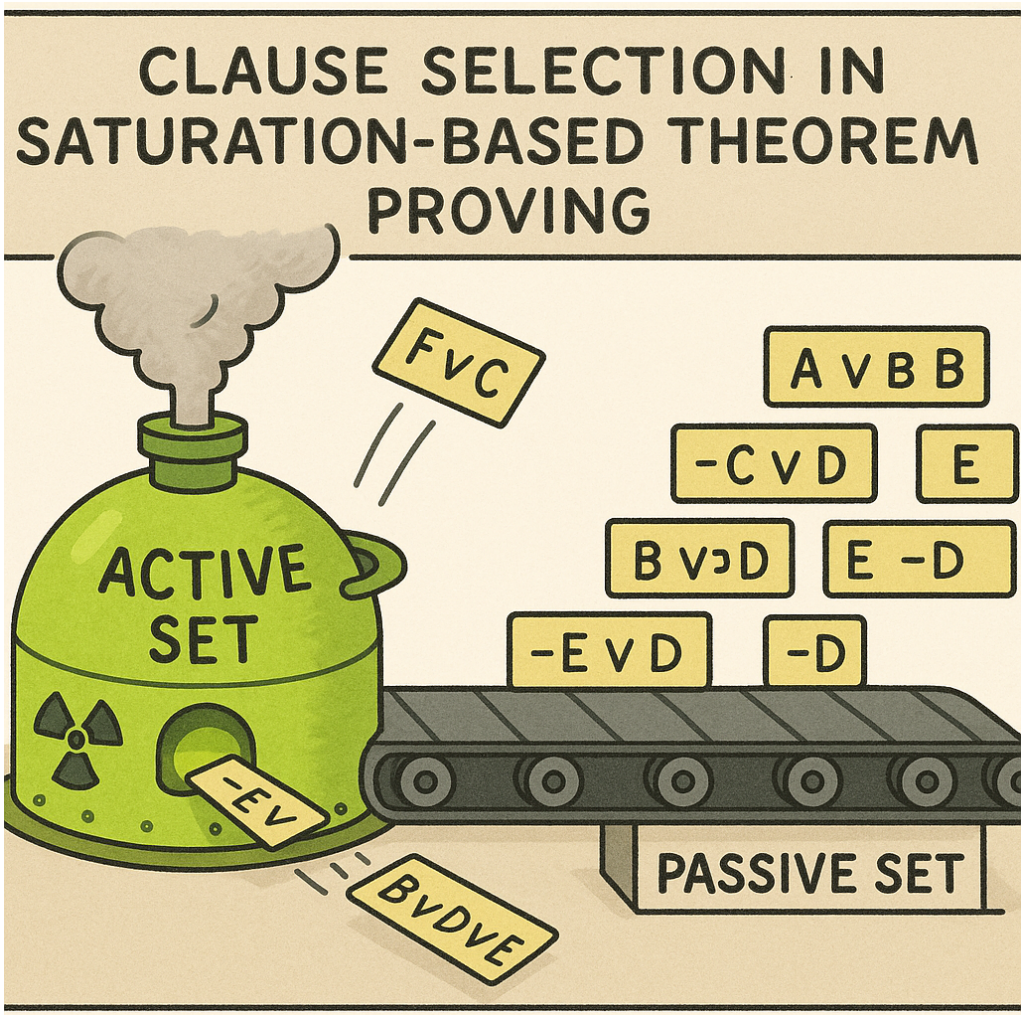
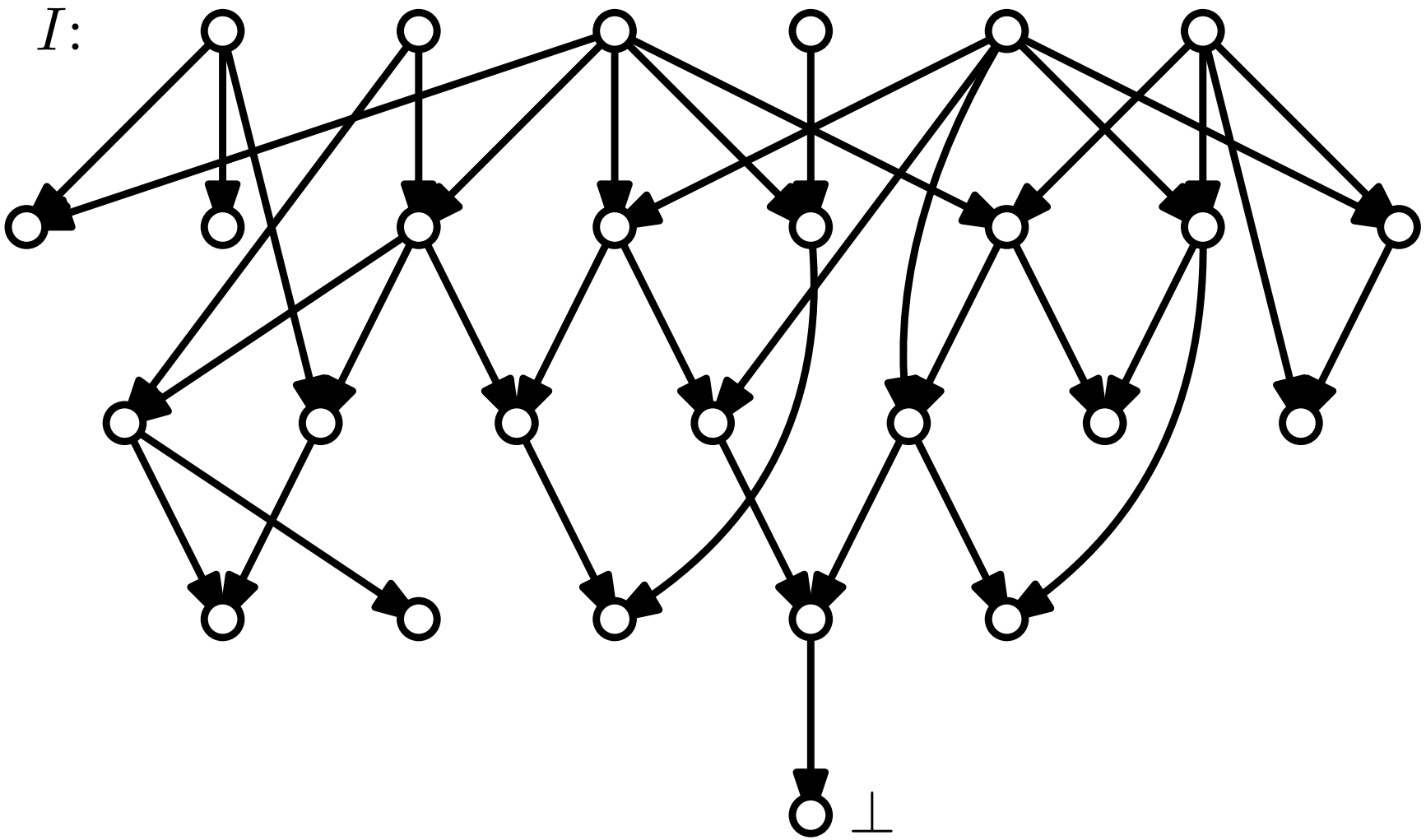


At a typical successful end: $|Passive| \gg |Active| \gg |Proof|$

The Proof Is Often Just A Tiny Part



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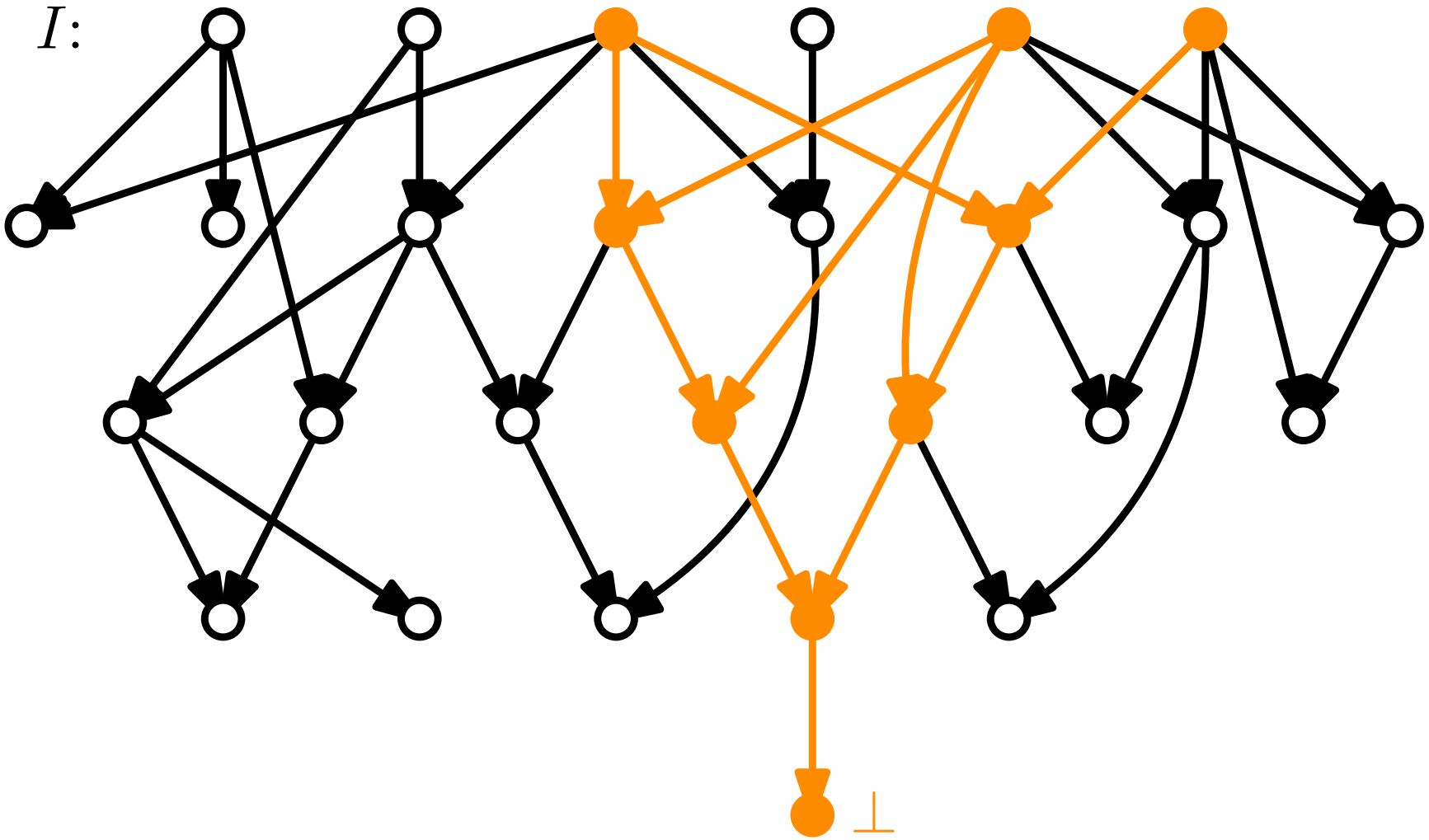

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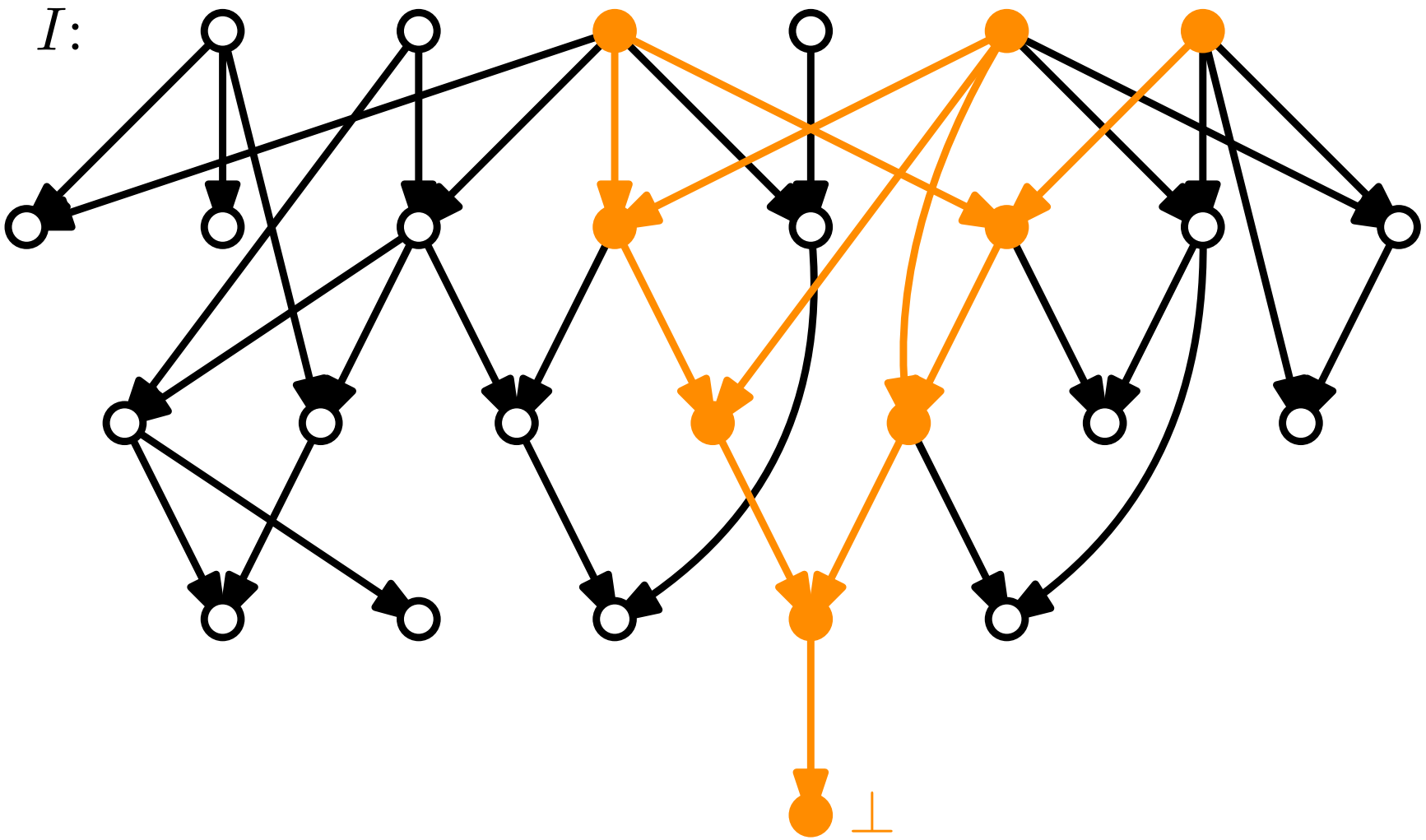

How close can we actually hope get to the perfect clause selection?

How is clause selection traditionally done?

Take simple clause evaluation criteria:

- age: prefer clauses that were generated long time ago
- weight: prefer clauses with fewer symbols

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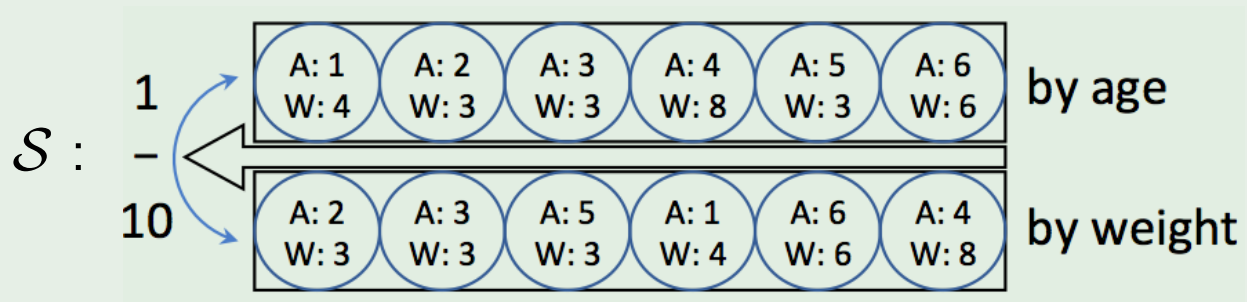
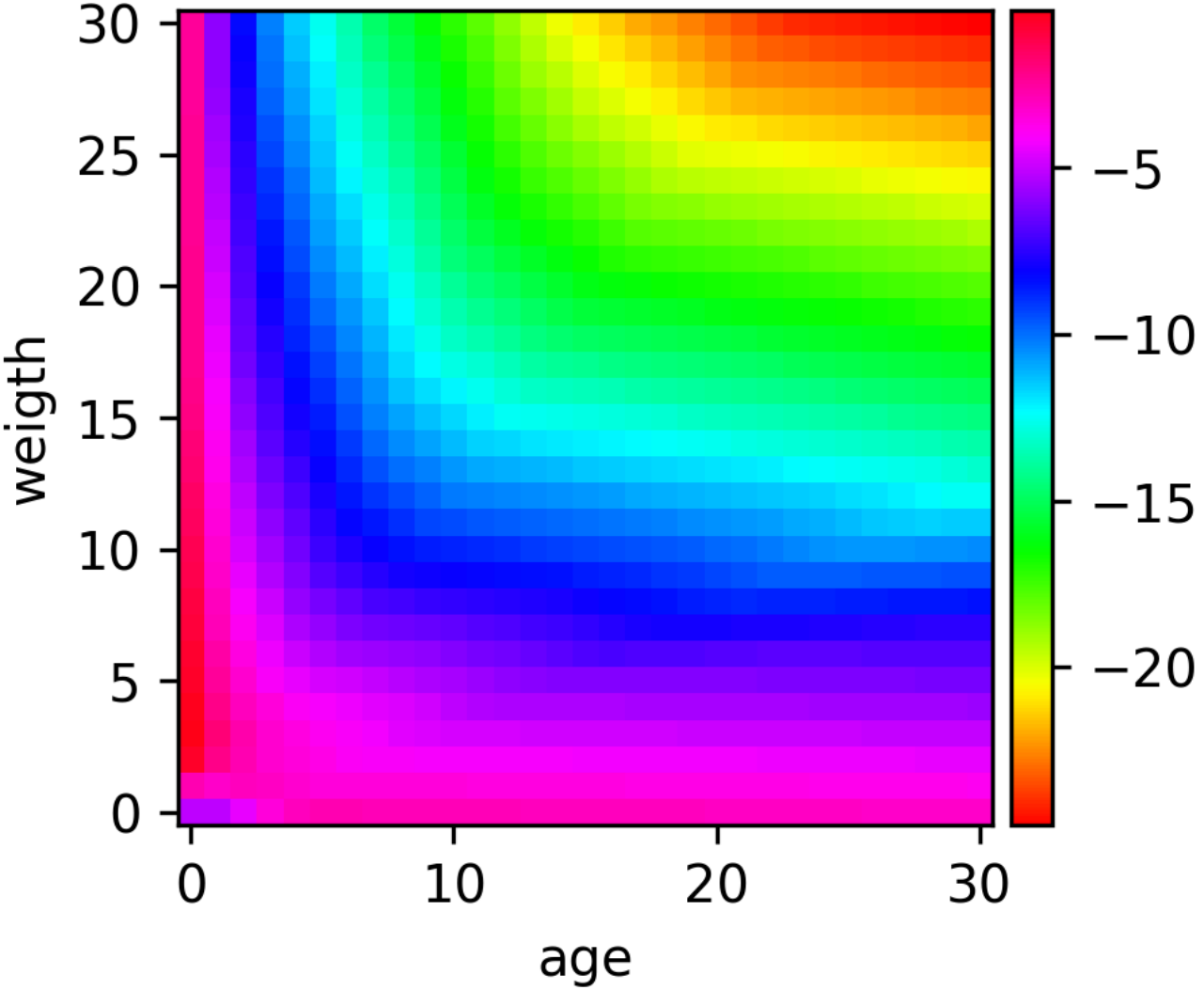
Example (Organizing *Passive* via two priority queues)



Sneak Peek: What Do NNs Think of Age and Weigth?



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What's really unique about RL?

- It programs itself (sometimes even optimally, in the limit)
- It could discover fundamentally novel tricks and hacks!

Key Reinforcement Learning Concepts

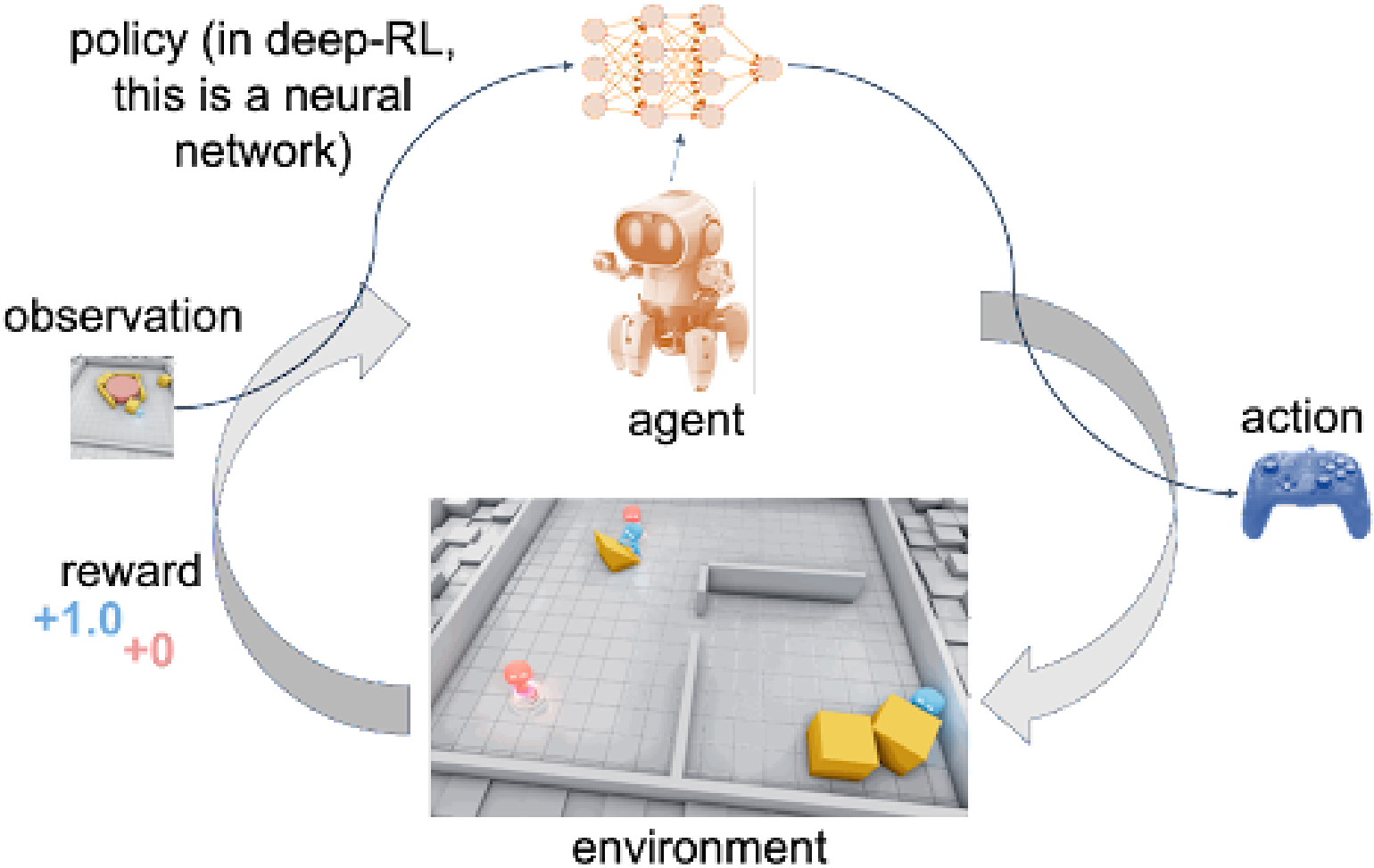

*Illustration from anyscale.com.

Saturation as an Reinforcement-Learning Environment

Agent

- the clause selection heuristic

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➥ TRAIL [Crouse et al.'21], [McKeown'23], [Shminke'23], ...

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Reward

- refusing the play the honest, super-sparse reward game
- like in ENIGMA: a proof clause is a good clause

Towards the RL-Inspired Learning Operator

A trace of a successful proof attempt on problem P is a tuple

$$T = (P, \mathcal{C}, \mathcal{C}^+, \{\mathcal{P}_i\}_{i \in I_T}).$$

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Logits and Policy

Assuming $N_{\boldsymbol{\theta}}$ produces a score $N_{\boldsymbol{\theta}}(C) = l_C$ for each clause C, then

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$$\pi_{C,\boldsymbol{\theta}} = \mathrm{softmax}_C\left(\{l_D\}_{D \in \mathcal{P}}\right) = \frac{e^{l_C}}{\sum_{D \in \mathcal{P}} e^{l_D}}$$

is the (stochastic) clause selection policy defined by $N_{\boldsymbol{\theta}}$

The RL-Inspired Operator

Policy Gradient Theorem [Williams'92]

To improve a policy in terms of the expected return we update

$$\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha r_C \nabla_{\boldsymbol{\theta}} \log \pi_{C,\boldsymbol{\theta}},$$

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Our Operator:
Each moment in time i is an independent opportunity to improve, with

$$\delta_i^T = \mathrm{mean}_{C \in \mathcal{P}_i^+} \nabla_{\boldsymbol{\theta}} \log \pi_{C,\boldsymbol{\theta}},$$

for a trace $T = (P, \mathcal{C}, \mathcal{C}^+, \{\mathcal{P}_i\}_{i \in I_T})$ and $\mathcal{P}_i^+ = \mathcal{P}_i \cap \mathcal{C}^+$. Then

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Aim for a balance between expressivity and speed of inference!

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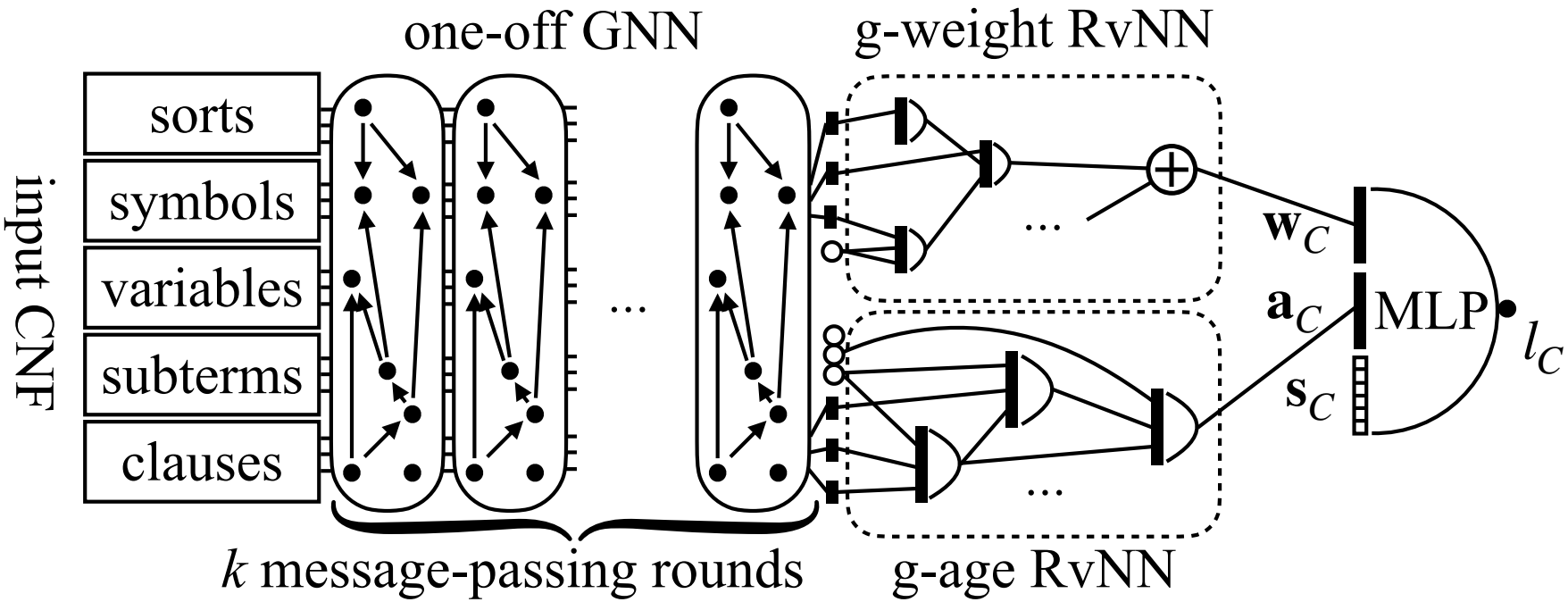
Simple Hand-Crafted Features on Top!

Architecture Diagram



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                parents:list[int]):                                  3
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              1 + max(height[p] for p in parents))                   5
  height[cl_num] = level                                             6
  index = level - base_level                                         7
  if len(todo_layers) == index:                                      8
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I still need to try out how much GPUs could help here . . .

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- little trick; despite the RL heritage:
  inner loop trains until validation loss does not improve

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- TPTP v9 CNF+FOF, 19 477 problems (train/test split)
- Vampire's default strategy (1:1 age-weight alternation)
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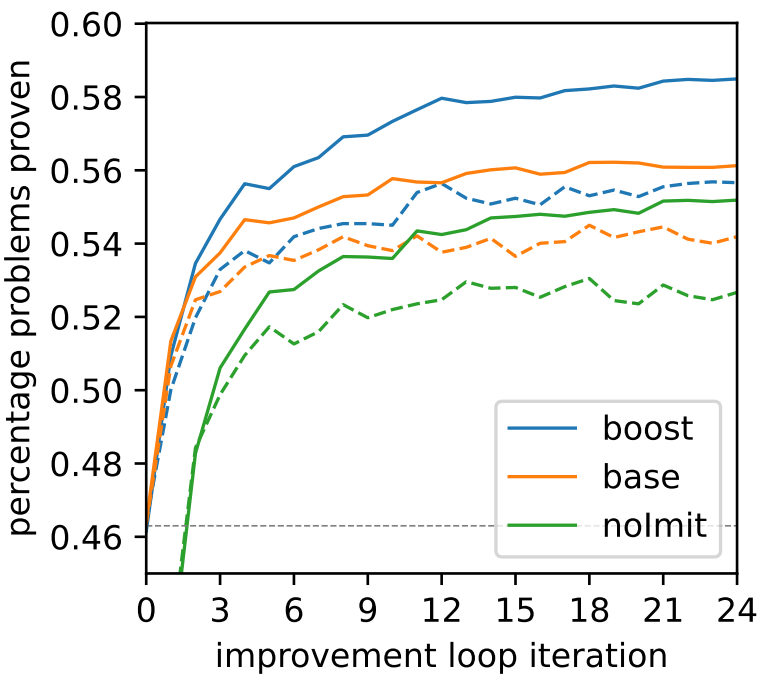

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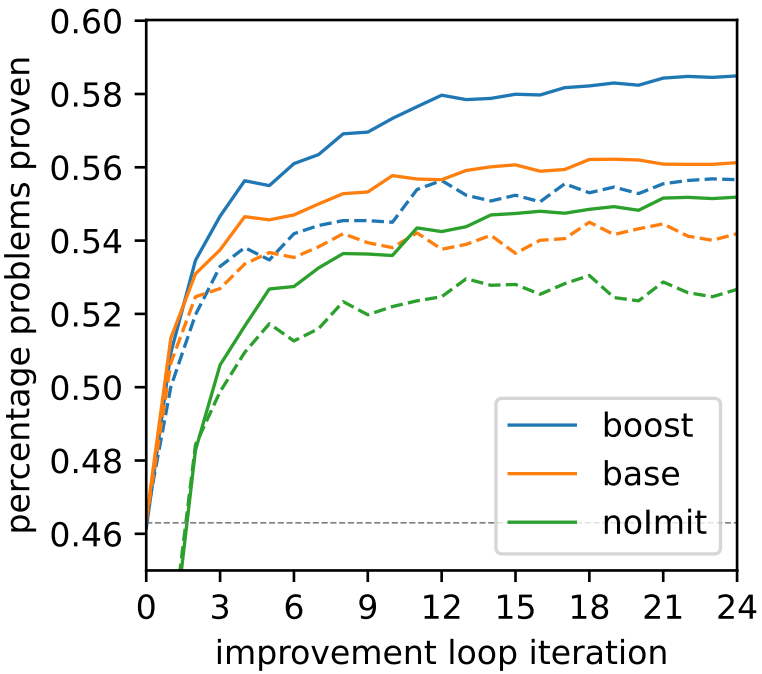

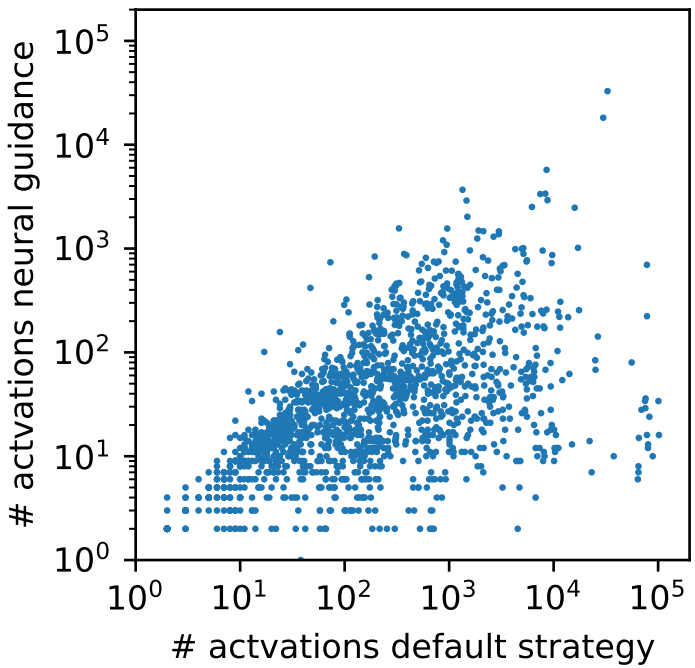

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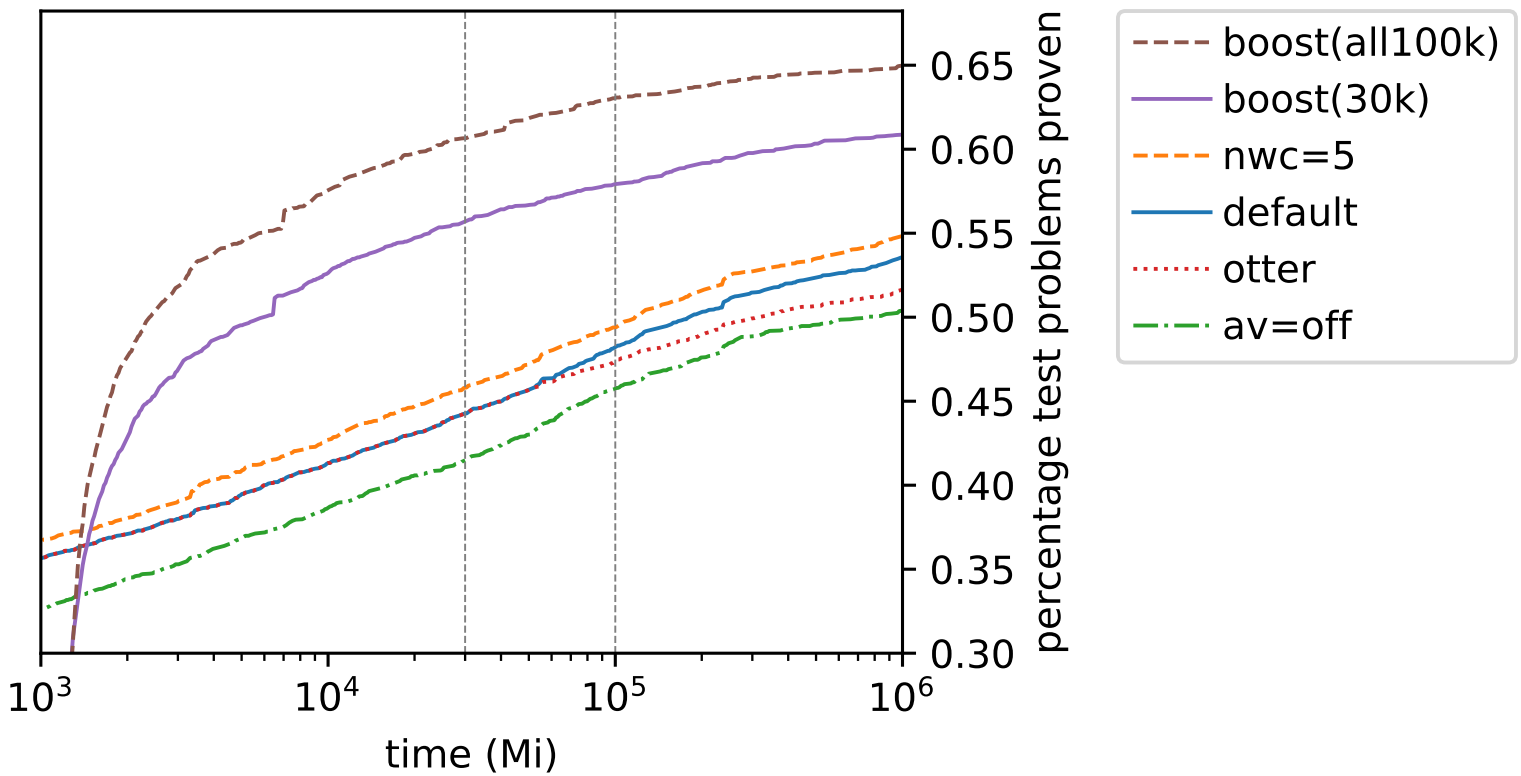

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- other benchmarks than TPTP; e.g. Mizar40; transfer learning
- neural guidance and theorem proving strategies

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PhD & PostDoc Position Open!

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The core idea

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Combine with the original strategy

$\mathcal{S} \oplus \mathcal{M}^{1,0}$: